ABELIAN p-GROUPS AND THE HALTING PROBLEM

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ABSTRACT. We investigate which effectively presented abelian p-groups are isomorphic relative to the halting problem. The standard approach to this and similar questions uses the notion of Δ_2^0-categoricity (to be defined). We partially reduce the description of Δ_2^0-categorical p-groups of Ulm type 1 to the analogous problem for equivalence structures. Using this reduction, we solve to a problem left open in [5]. For the sake of the mentioned above reduction, we introduce a new notion of effective Δ_2^0-categoricity that lies strictly in-between plain Δ_2^0-categoricity and relative Δ_2^0-categoricity (to be defined). We then reduce the problem of classifying effective Δ_2^0-categoricity to a question stated in terms of Σ_2^0-sets. Among other results, we show that for c.e. Turing degrees bounding such sets is equal to being complete.

1. INTRODUCTION

Following Mal'cev [22] and Rabin [24], we say that an algebraic structure is *computable* or *constructive* if there exists an numbering of its elements by natural numbers under which the operations, relations and equality become Turing computable. This numbering is called a computable presentation or constructivization of the structure. For example, a group has a computable presentation if and only if it has a "recursive presentation" (Higman [16]) with decidable word problem. This definition also generalizes the early notion of an "explicitly presented" filed due to van der Waerden [27] (formally clarified by Fröhlich and Shepherdson [11]).

The general philosophy of effective algebra is that effectively presented objects should be studied under effective isomorphisms. Following the standard terminology [1, 10], we say that a computable algebraic structure is *computably categorical* or *autostable* if every two computable presentations of the structure agree up to a computable isomorphism. Most nontrivial "natural" examples of computable algebraic structures are *not* computably categorical. For example, only very few abelian p-groups [26] are computably categorical, and those are trivial; see [15, 10, 1] for more examples. This paper contributes to a general framework (e.g., [2, 23, 3, 9, 6]) that investigates computable structures which are *not* computably categorical but are *close* to being computably categorical (to be explained).

In contrast to computably categorical structures that are rare, computable structures that are isomorphic *relative to* the halting problem $0'$, or maybe relative to a few iterations of the halting problem, often occur in mathematical practice[1]. That is, if we had an oracle for $0^{(n)}$, we could compute an isomorphism. Intuitively, it means that to build an isomorphism it is sufficient to understand only a few alternations of quantifiers over a computable relation [25]. Indeed, we typically use at most $0'''$-injury techniques when we construct two or more different computable presentations of an algebraic structure. As a consequence, unless there is a pattern that we could iterate, the isomorphisms that we can handle are usually at most $0'''$. An elementary example of this phenomenon is the classical Mal'cev's construction of a $\mathbb{Q}$-vector

[1]As usual, $0^{(n+1)}$ stands for the n'th iteration of the halting problem, up to Turing equivalence. More generally, the *Turing jump operator* $X \to X'$ is well-defined up to Turing equivalence on arbitrary oracles $X \subseteq \mathbb{N}$.

space in which linear independence is undecidable [22]. The standard "nice" and the Mal'cev's "complicated" presentations are isomorphic relative to the halting problem $0'$. In fact, *any* two computable copies of this vector space are isomorphic relative to $0'$. A non-elementary example is a remarkable result of Goncharov, Molokov and Romanovskiĭ [14] (based on Goncharov [13]) saying that there exists a computable, infinitely generated nilpotent group with *exactly two* computable presentations up to computable isomorphism. This bizarre nilpotent group has a unique computable representation up to $0''$-isomorphism. It is not known whether this upper bound on the complexity of isomorphism could be improved to some $\mathbf{a} <_T 0''$. It is known, however, that if any two computable presentations of a structure are $0'$-isomorphic, then the structure has infinitely many computable presentations up to computable isomorphism [12, 10]. For instance, many abelian groups have this property [12]. We refer to [1, 10, 17] for more examples of this nature.

Seeking a deeper understanding of these and similar constructions, we would like to accumulate more knowledge about computable structures isomorphic relative to a few iterations of the halting problem. The definition below was suggested by Ash.

Definition 1.1. A computable algebraic structure A is Δ_n^0-categorical if every two computable presentations of A are $\emptyset^{(n-1)}$-isomorphic.

Clearly, Δ_n^0-categoricity is a natural generalization of computable categoricity (set $n = 1$), and thus the notion is interesting on its own right. Ash [2] was the first to systematically study Δ_n^0-categorical computable structures. He described Δ_n^0-categorical well-orders. Although there are several further deep results on Δ_n^0-categorical structures in the literature ([3, 23, 8], see also Chapter 17 of [1]), our understanding of Δ_n^0-categoricity is rather limited even when $n = 2$. While computable categoricity was characterized for Boolean algebras, linear orders, torsion-free ablelian groups and many other standard classes ([1, 10]), we don't have a satisfactory description of Δ_2^0-categoricity in any of these classes. As it seems, Δ_2^0-categoricity is far less well-behaved than computable categoricity. For instance, in contrast to computable categoricity, Δ_2^0-categoricity tends to be different from *relative Δ_2^0-categoricity*[2] already in rather simple algebraic classes [4, 18, 5]. As a consequence, the study of Δ_2^0-categoricity usually requires new algebraic and computability-theoretic ideas (e.g., [9, 18]), and thus such investigations are of some technical interest as well.

1.1. **Complex isomorphisms between simple structures.** Our intention is to study Δ_2^0-categoricity and Δ_2^0-isomorphisms within an algebraic context which is as simple as possible. We would like to pick a class where algebra would not be the main obstacle (in contrast to, say, [23, 9]) and concentrate on the *computability-theoretic* combinatorics of Δ_2^0-isomorphisms. At the same time we wish Δ_2^0-categoricity to have a non-trivial behavior in the class that we chose. Calvert, Cenzer, Harizanov, and Morozov [5, 4] discovered two classes that perfectly meet these requirements (as we will see). These are the classes of computable equivalence structures and of abelian p-groups having Ulm type 1 (to be defined).

In this context, an equivalence structure is an abstraction to the situation when a computable structure has several "components". For example, think of a direct or cardinal sum of groups or rings, or imagine a graph having several connected components. We remove all structure from each component and concentrate on the fundamental property:

> From stage to stage, each component can only grow in size.

To build an isomorphism, we at least need to match the sizes of components correctly.

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[2]Recall that a computable structure $\mathcal{M}$ is relatively Δ_n^0-categorical if the $(n-1)'th$ Turing jump $D_0(\mathcal{A})^{(n-1)}$ of the open diagram $D_0(\mathcal{A})$ of $\mathcal{A} \cong \mathcal{B}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Note $\mathcal{A}$ does not have to be computable.

The reader not familiar with the subject will perhaps be surprised that *we still don't know which equivalence structures are Δ_2^0-categorical.* It might be even more surprising that the study of Δ_2^0-categorical equivalence structures involves deep computability-theoretic methods such as $0'''$-techniques, see satellite paper [7]. The difficulties are rooted in the existence of Δ_2^0-categorical computable equivalence structures that are not relatively Δ_2^0-categorical [4, 18].

In the paper in hand we investigate abelian p-groups where things get even harder. When we look at abelian p-groups of Ulm type 1, we essentially add just a little algebra to make the components non-stable up to automorphism. As a consequence, we have to worry about algebra *as well*, but the algebra is relatively tame.

1.2. **Results.** In [5], Calvert, Cenzer, Harizanov and Morozov attempted to describe Δ_2^0-categoricity for the *classically* elementary class of non-reduced abelian p-groups having Ulm type 1. These are direct sums of cyclic and quasi-cyclic (Prüfer) p-groups. Thus, a typical member of this class is of the form

$$\bigoplus_{k\in S}(\bigoplus_{i\in S_k}\mathbb{Z}_{p^k})\oplus\bigoplus_{j\in J}\mathbb{Z}_{p^\infty},$$

where $\mathbb{Z}_{p^\infty}$ is the Prüfer group, $S, J \subseteq \omega$ and $S_k \subseteq \omega$ for each k. For convenience, we call such abelian p-groups *multicyclic* groups. Calvert et. al. [5] obtained several sufficient conditions for a computable multicyclic group to be Δ_2^0-categorical. They left wide open the following question:

> *Which computable multicyclic groups are Δ_2^0-categorical?*

In fact, they asked two questions that are more specific; we will answer one of these two questions, the other is answered in [7].

A multicyclic group is "almost" an equivalence structure, in the following sense. Given a multicyclic A, let E_A be an equivalence structure having classes reflecting the sizes of elementary summands in A. Observe that A is computably presentable if, and only if, E_A is computably presentable. Unfortunately, it is not the case that an abelian group A is Δ_2^0-categorical if, and only if, the equivalence structure E_A is Δ_2^0-categorical, as one might expect. More specifically, it follows from [5] that there exist multicyclic groups with exactly one Prüfer summand that are *not* Δ_2^0-categrical.

Nonetheless, under some extra effectiveness and algebraic conditions we *can* get a reduction to equivalence structures. To obtain such a reduction, we need a new notion of categoricity.

Definition 1.2. Say that a computable algebraic structure is *effectively Δ_2^0-categorical* if there exists an effective procedure which, given indices i, j of computable copies M_i, M_j of the structure, produces a Σ_2^0-index e for a Δ_2^0-isomorphism from M_i onto M_j.

Clearly, we have

$$\text{relative } \Delta_2^0\text{-categoricity} \Rightarrow \text{effective } \Delta_2^0\text{-categoricity} \Rightarrow \text{plain } \Delta_2^0\text{-categoricity},$$

and we will see all these implications are strict. The algebraic condition is:

Definition 1.3. We say that a multicyclic group is *degenerate* if either the orders of its cyclic summands are bounded, or it has only finitely many Prüfer summands (or both).

Degenerate multicyclic groups are of no interest for us, since a degenerate mulitcyclic group is Δ_2^0-categorical if and only if either its has no Prüfer summands or its cyclic summands are bounded [7, 5].

The first main result of the paper is:

Theorem 3.7. *Let A be computable non-degenerate multicyclic group. Then A is effectively Δ_2^0-categorical if, and only if, the corresponding equivalence structure E_A is effectively Δ_2^0-categorical.*

Calvert, Cenzer, Harizanov and Morozov [5], using a different terminology, asked

Can a non-degenerate multicyclic group be Δ_2^0-categorical?

As we will see, it is not difficult to construct an equivalence structure that is effectively Δ_2^0-categorical. As a consequence of Theorem 3.7, we answer the above question in positive. In fact, Theorem 3.7 and its corollaries provide us with an infinite family of such groups. Our proof of Theorem 3.7 heavily relies on the specific uniform properties of effective Δ_2^0-categoricity. We don't know if the result can be generalized to plain Δ_2^0-categoricity, and we suspect that the answer could be "no". The proof of Theorem 3.7 also uses a technical result (Theorem 3.1) that expresses (effective) Δ_2^0-categoricity of a multicyclic group in terms of its p-height function. Theorem 3.1 has some independent interest since its proof gives a necessary and sufficient condition for a computable multicyclic group to be completely decomposable effectively in an oracle X. Such studies are not very common for computable p-groups, but a lot more has been done for torsion-free abelian groups [20, 21, 8]. We also note that the reader may find the proof of Theorem 3.1 unexpectedly involved. There are two reasons for this complexity. Firstly, we could not really refer to any classical result in abelian group theory since the group is not reduced, and classically detaching the divisible part is the first thing one would do. Secondly, the dynamic process of detaching the divisible part is the core of the proof since it has to be done in the most "economical" way; this is something we never see in classical algebraic texts.

1.3. **Effective Δ_2^0-categoricity.** Theorem 3.7 completely separates effective Δ_2^0-categoricity of multicyclic groups from the group-theoretic context and reduces the question to equivalence structures. We thus concentrate on the study of effectively Δ_2^0-categorial equivalence structures.

In Theorem 2.5, we show that effective Δ_2^0-categoricity is not dependent on repetitions of finite classes. That is, it does not matter whether we have one, two, $k \in \omega$ or infinitely many classes of some fixed size. We can just keep exactly one class of each finite size that occurs in the structure. We find ourselves in the following situation. For a set $X \subseteq \omega$, let $E(X)$ be an equivalence structure having infinitely many infinite classes and exactly one class of size x, for each $x \in X$. Note that X is Σ_2^0 if, and only if, $E(X)$ has a computable copy. We say that a Σ_2^0-set is effectively categorical if $E(X)$ is effectively Δ_2^0-categorical. Thus, we further reduce the situation to a problem for Σ_2^0-sets:

Which Σ_2^0-sets are effectively categorical?

The second main result of the paper describes c.e. Turing degrees bounding effective categoricity:

Theorem 1.4. *A c.e. degree $\boldsymbol{a}$ bounds an infinite effectively categorical set if, and only if, $\boldsymbol{a}$ is complete.*

Theorem 1.4 follows from Theorem 4.2 which says more. More specifically, we isolate a recursion-theoretic property which captures effective categoricty of a set. We call this property *strong dominance*. In the context of c.e. degrees, it guarantees completeness, but not in general (Corollary 4.11). The reader may compare Theorem 1.4 with the second main result of the satellite paper [7]. Sections 2 and 4 contain some further results on effectively categorical Σ_2^0-sets which we do not state here.

We organize the paper as follows. In Section 2, we accumulate some basic knowledge about effectively Δ_2^0-categorical equivalence structures. In Section 3, we prove Theorems 3.1 and 3.7. In Section 4, we prove the second main result of the paper, and also obtain some further results on effectively categorical sets. A further discussion can be found in Section 5.

2. EFFECTIVE Δ_2^0-CATEGORICITY

Let $(\psi_i)_{i \in \omega}$ be the effective listing of all partial $0'$-computable functions from ω to $\omega \cup \{\infty\}$ obtained from the limit lemma. We say that i is a Σ_2^0-*index*, or a Δ_2^0-*index*, or simply an *index* for the partial function represented by ψ_i. Recall that a computable algebraic structure is effectively Δ_2^0-categorical if there exists an effective procedure which, given indices i, j of computable copies M_i, M_j of the structure, produces a Σ_2^0-index e for a Δ_2^0-isomorphism from M_i onto M_j.

2.1. **Effective Δ_2^0-categoricity of equivalence structures.** We use $\#[i]$ to denote the size of a class $[i]$ in an equivalence structure. We call $\#$ the size function. We also say that an equivalence structure is *nondegenerate* if it has infinitely many infinite classes, and sizes of finite classes are unbounded.

Proposition 2.1. For a computable, nondegenerate equivalence structure E, the following are equivalent:

(1) E is effectively Δ_2^0-categorical.
(2) There exists a uniform procedure which, given an index of a computable copy of E, returns a Δ_2^0-index for the size function $\#$ in that copy.

Proof. Observe that every non-degenerate computable equivalence structure has a *regular* copy in which $\#$ is Δ_2^0. It is sufficient to instantly grow a class representing x to infinity if our guess changes to $x \notin X$. We then reintroduce the size x to the structure, if our guess changes to $x \notin X$. It remains to apply the definition of effective Δ_2^0-categoricity. $\square$

Each infinite Σ_2^0 set corresponds to a computably presentable equivalence structure $E(X)$ having infinitely many infinite classes and exactly one class of size n for $n \in X$.

Definition 2.2. An infinite Σ_2^0 set X is *effectively categorical* if $E(X)$ is effectively Δ_2^0-categorical.

The following lemma gives a necessary and sufficient condition for a set to be effectively categorical. This condition is much more convenient than Definition 2.2 in applications.

Lemma 2.3. For a Σ_2^0 set X, the following are equivalent:

(1) X is effectively categorical;
(2) there is a (partial) Δ_2^0-function $g : \omega \to \omega \cup \{\infty\}$ such that for every total non-decreasing function φ_e,

$$g(e) = \begin{cases} \infty, & \text{if } ran(\varphi_e) \text{ is infinite,} \\ f, & \text{if } ran(\varphi_e) \text{ is finite and } \max ran(\varphi_e) \in X. \end{cases}$$

(If $\max ran(\varphi_e) \notin X$, then $g(e)$ may be either undefined or equal to any value.)

Proof. (1) $\Rightarrow$ (2). Suppose that X is effectively Δ_2^0-categorical. Given e, produce an equivalence structure R_e as follows. Create an equivalence class of size $\max ran(\varphi_e)$, and adjoin this class to either a copy of $E(X)$, if $\max ran(\varphi_e)$ is not finite or is not in X, or to a copy of $E(X \setminus \max ran(\varphi_e))$, otherwise. We can produce R_e effectively in e by making a class grow to infinity if it appears to be equal to $\max ran(\varphi_{e,s})$ at stage s, and introducing the corresponding size later if necessary. There is a total computable function h such that $R_e = M_{h(e)}$. There is a

(total) computable function f which, given $M_e \cong E(X)$, outputs an index for an isomorphism from M_e onto the regular copy of $E(X)$ produced in the proof of Proposition 2.1. Recall that this regular copy already has a Δ_2^0-function $\hat{g}$ which represents $\#$. Define the desired g by the rule $g(e) = \hat{g}\Phi_{fh(e)}(0)$.

$(2) \Rightarrow (1)$. Suppose that such a function g exists. Given e, define a computable function h such that $\varphi_{h(n)}$ is total and

$$\max ran(\varphi_{h(n)}) = \begin{cases} \#C_n, & \text{if } M_e = \{C_n\}_{n\in\omega} \text{ is a computable equivalence structure,} \\ \infty, & \text{otherwise,} \end{cases}$$

for every n. The Δ_2^0-index for h and hence $g \circ h$ can be obtained from e uniformity. If $M_e \cong E(X)$, then we can use $g \circ h$ to build an isomorphism from E to the "regular" representation of $E(X)$ (produced in the proof of Proposition 2.1) uniformly in $g \circ h$. $\qquad\square$

Remark 2.4. In Lemma 2.3, the Δ_2^0-function $g : \omega \to \omega \cup \{\infty\}$ can be equivalently replaced by a (partial) Δ_2^0-function $\hat{g} : \omega \to \{f, \infty\}$, where f indicates that the domain is finite.

Given a computable equivalence structure R, keep only one class of each finite size represented in R. Denote the resulted structure $\widehat{R}$ and call it *the condensation of R*. Notice that $\widehat{R}$ is computably presentable, since the collection of finite sizes that occur in $\widehat{R}$ is a Σ_2^0 set.

Theorem 2.5. *A computable, nondegenerate equivalence structure R is effectively Δ_2^0-categorical if, and only if, $\widehat{R}$ is effectively Δ_2^0-categorical.*

Proof. To prove that effective Δ_2^0-categoricity of $\widehat{R}$ implies effective Δ_2^0-categoricity of R, combine the s-m-n Theorem, Proposition 2.1, and Lemma 2.3. For the converse, consider the Σ_2^0 set

$$\{\langle m, k \rangle : \text{ there are at least } k + 1 \text{ many classes of finite size } m \text{ in } R\}.$$

This set corresponds to an equivalence structure with infinitely many infinite classes having a computable copy V. Given a computable copy X of $\widehat{R}$, take a disjoint union Y of X to V. The resulting computable structure is a computable copy of R, and thus has a $0'$-computable function guessing sizes in Y correctly. Since the operation of taking the disjoint union is effective, an index for the restriction of this function to the domain of X can be obtained effectively from the index for X. $\qquad\square$

3. Categoricity of groups

In this section all groups are countable abelian p-groups, where p is any fixed prime number. Recall that a non-trivial cyclic p-group is isomorphic to $(\mathbb{Z}_{p^m}, +)$ for some $m \in \omega$, and recall that the quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$ is the direct limit of the sequence

$$0 \to Z_p \to \mathbb{Z}_{p^2} \to \dots,$$

under the natural identity embeddings. A group is *multicyclic* if it is countable and is isomorphic to a direct sum of cyclic and quasi-cyclic p-groups.

For an abelian p-group A, the p-height $\mathrm{h}_p^A(a)$ of a non-zero $a \in A$ is the maximal $m \in \omega$ such that $(\exists b \in A)\, p^m b = a$ if such an m exists, and $\mathrm{h}_p^A(a) = \infty$ otherwise. The p-height of 0 is ∞. In the following, we suppress p in the term p-height, and we typically omit A in $\mathrm{h}_p^A(a)$ if it is clear from the context what A is. A subgroup V of an abelian p-group A is *pure* if for every $v \in V$

$$\mathrm{h}_p^V(v) = \mathrm{h}_p^A(v).$$

Direct summands are always pure, but not every pure subgroup is a direct summand. Nonetheless, it is well-known that every pure cyclic subgroup of an abelian group detaches as a summand. The same holds for a quasi-cyclic subgroup (which is always pure).

The result below describes computable (effectively) Δ_2^0-categorical multicyclic groups in terms of the complexities of their height function.

Theorem 3.1. *Let A be a computable multicyclic group.*

    (1) A is Δ_2^0-categorical if, and only if, $\mathtt{h}_p \leq 0'$ in every computable presentation of A.

    (2) A is effectively Δ_2^0-categorical if, and only if, there exist a uniform procedure computing a Δ_2^0 index for $\mathtt{h}_p \leq 0'$ in a computable copy of A from an index of the computable copy.

Proof. We prove (1), and then for (2) we observe that the proof of (1) is sufficiently uniform.

Suppose A is Δ_2^0-categorical. Similarly to computable equivalence structures, there exists a computable copy B of A in which $\mathtt{h}_p$ changes at most once, with essentially the same proof. Given any computable copy C of A and an isomorphism $\psi : C \to B$, we can determine $\mathtt{h}_p(c)$ taking the minimum of heights of elements b_i in the decomposition $\psi(c) = \sum_i m_i b_i$, where $m_i \in Z_p$ and b_i are taken from different elementary components of B.

Suppose $\mathtt{h}_p \leq 0'$ in every computable presentation of A. We fix a computable C isomorphic to A. It is sufficient to find, effectively in $\mathtt{h}_p$, a complete decomposition $\bigoplus_i V_i$ of C into cyclic and quisi-cyclic summands. Once such a decomposition is found, we could take the "regular" copy B of A in which such a decomposition is already known, and then map C onto B component-wise according to the heights.

Informal discussion. Note that $C = R \oplus U$, where U is a divisible group and R is reduced. Furthermore, U is an $\mathtt{h}_p^C$-computable subgroup of C, and R is isomorphic to a direct sum of cyclic groups. We of cause have no access to R, since it is not an invariant subgroup. On the other hand, if we have $r + u$, where $r \in R$ and $u \in U$, then u has no effect on the height of $r + u$. Thus, if at a stage we have enumerated finitely many cyclic summands and have initiated enumeration of finitely many quasi-cyclic summands, then the height of a new chosen element in the factor-group C/U can be effected (when compared to $\mathtt{h}_p^C$) only by those finite cyclic summands we already have. Since these finite cyclic summands are uniformly effective in $\mathtt{h}_p^C$, we will conclude that the whole decomposition is computable uniformly in $\mathtt{h}_p^C$.

Notations. In the following, $\{c_i : i \in \omega\}$ stands for an effective listing of C. We may assume $c_0 = 0$, but it has no effect on the uniformity of the proof, since 0 can be effectively found. In the following, the parameter k_s is counting the number of elementary summands whose enumerations have been initiated. If we do not introduce any new summand at stage s, then we set $k_s = k_{s-1}$.

Construction. At stage 0, set $V_0 = \{0\}$ and $k_0 = 0$. At stage s, suppose we have already defined a finite sequence $V_0, \dots V_{k_{s-1}}$ of cyclic and finite initial segments of quasi-cyclic groups. Let $D_s = \sum_{i \leq k_{s-1}} V_i$. Stage s has substages I and II:

I. Expanding D_s. Let i be least such that V_i contains only elements of infinite height, and V_i which has not yet been expanded at stage s. If there is no such V_i which needs to be extended, then proceed to the next substage described below. Otherwise, suppose that $v_0, \dots, v_{k_{s-1}}$ are the generators of $V_0, \dots, V_{k_{s-1}}$, respectively. Find an element u such that:

    (1) $pu = v_i$;

    (2) $\mathtt{h}_p(u) = \infty$;

    (3) the sum $\langle u \rangle + \sum_{i \neq j} \langle v_i \rangle$ is direct.

Set $v_i = u$ and proceed to the next least j such that V_j needs to be expanded, if there are any left.

II. *Introducing new summands.* Let i be least such that $x = c_i$ satisfies $x \notin D_s$ and $px \in D_s$.

Case 1: $\mathtt{h}_p(x) = \infty$.

    − if $px = 0$, then set $v_{k_s} = x$ and $V_{k_s} = \langle x \rangle$.

    − if $px = u \neq 0$, then delay the action until the next substage "expanding D_{s+1}" and check if x is in D_{s+1} at the end of this substage. If yes, do nothing. If no, then take a $v \in D_s$ such that $pv = px$, set $v_{k_s} = x - v$, and $V_{k_s} = \langle x - v \rangle$.

Case 2: $\mathtt{h}_p(x) = m < \infty$. If $\sup_{d' \in D_s} \mathtt{h}_p^C(x + d') = \infty$, then do nothing. Otherwise, take $d \in D_s$ such that

$$\mathtt{h}_p^C(x + d) = \sup_{d' \in D_s} \mathtt{h}_p^C(x + d') = m'$$

and set $x' = x + d$. Find $a \in C \setminus D_s$ such that $m'a = x'$.

    − if $px' = 0$, then set $v_{k_s} = a$ and $V_{k_s} = \langle a \rangle$.

    − if $px' = u \neq 0$, then, possibly delaying the substage, wait for a y such that $p^{m'+1}y = u$ to be enumerated into D_t at substage I of $t \geq s$. If $x' \notin D_s$, then set $v_{k_t} = a - y$ and $V_{k_s} = \langle a - y \rangle$.

Verification. Note that every search at every substage can be done effectively and uniformly in $\mathtt{h}_p$. (We should still verify that every substage can successfully finish its search.) Given a parameter f, we write $f[s]$ for its value at stage s.

Lemma 3.2. For every s, $D_s = \bigoplus_{i \leq k_{s-1}} V_i$.

Proof. Note that at every stage we maintain

$$V_i \cap \sum_{j \neq i, j \leq k_s} V_i = \mathbf{0},$$

which is a necessary and sufficient condition for the sum to be direct. $\square$

Lemma 3.3. For every s the substage *Expanding D_s* of stage s succeeds in extending the already existing groups V_i, $i \leq k_{s-1}$, having non-zero elements of infinite height.

Proof. Take any u satisfying (1) and (2) of the substage. We show that there exists a u' satisfying (1), (2), and (3). Consider $\{x : px = 0; \mathtt{h}_p(x) = \infty\} \leq C$ which is a Z_p-vector space of infinite dimension, and let $d', d'', d''', \ldots$ be an infinite basis of this space. By the assumption the group is non-degenerate, whence there exists a summand H of C of finite rank containing D_s and u but not containing $d^{(i)}$ for cofinitely many i. Thus, we may assume it is true for all of the $d^{(i)}$, under a change of notations. Then the $d^{(i)}$ are all Z_p-independent in C/H. Suppose that for every $d^{(i)}$ there exists a non-trivial linear combination

$$m_i(u + d^{(i)}) = \sum_j m_{j,i} v_j,$$

where v_j generates V_j at stage s (we omit [s] in $v_j[s]$ generates $V_j[s]$), and $1 \leq |m_i| < p$. We have

$$m_i(u + d^{(i)}) = 0 \bmod H$$

The equality above implies that $m_i d^{(i)} + m_j d^{(j)} = 0 \bmod H$ contradicting the choice of H, m_i, m_j and the elements $d^{(i)}$ and $d^{(j)}$. Assuming that D_s already satisfies:

$$V_i \cap \sum_{j \neq i, j \leq k_s} V_i = \mathbf{0},$$

we conclude that there exists a $d^{(i)}$ such that expanding V_i with using $u' = u + d^{(i)}$ will not violate the property above. $\qquad\square$

We need to show that the second substage can not be delayed forever and will be finished once we find a sufficiently large extension of D_s (see the construction). The reader may think of s in the lemma below as of a stage at which the "Introducing new summands" substage is delayed and waits for D_s to grow to a sufficiently large D_t having a chain witnessing some specific height. The algebraic meaning of the lemma is that every finite $\sum_{j \leq k} V_j[s]$ will eventually be extended to a summand of C, and this process does not depend on substages of the second form. The proof of the lemma also implies that we are never stuck in extending D_s to another finite cyclic summand, unless it is already spanned by D_s.

Lemma 3.4. *Let F_s be the sum of all elementary components of D_s in the decomposition $D_s = \bigoplus_{i \leq k_{s-1}} V_i$ that are spanned by elements having finite height in C. Then $C = N \oplus F_s$, for some N, and for every $x \notin D_s$,*

$$\mathrm{h}_p^{C/F_s}(x) = \sup_{f \in F_s} \mathrm{h}_p^C(x+f) = \sup_{f \in D_s} \mathrm{h}_p^C(x+f).$$

Proof. The proof is an induction on the number of summands in F_t and it uses the stages at which these summands were introduced in the construction. The base of induction, $F_t = \mathbf{0}$, is a triviality. So suppose F_t detaches as a summand:

$$C = F_t \oplus N'.$$

We prove

$$\mathrm{h}_p^{C/F_t}(x) = \sup_{f \in F_t} \mathrm{h}_p^C(x+f) = \sup_{f \in D_t} \mathrm{h}_p^C(x+f).$$

The second equality follows at once since a divisible element can not effect the height. For the first equality, let U be *any* minimal divisible subgroup properly containing all the summands of D_t consisting of elements of infinite height. We have

$$C = F_t \oplus U \oplus H,$$

because U is contained in N', is divisible, and whence detaches as its summand. Let

$$x = f + u + h$$

be the decomposition of x into projections.

We have $\mathrm{h}_p^{C/F_t}(x) = \mathrm{h}_p^H(h) = \mathrm{h}_p^C(h)$, because u does not effect the height, and since direct summands are pure. We also have

$$\mathrm{h}_p^C(x) = \inf\{\mathrm{h}_p^H(h), \mathrm{h}_p^{F_t}(f), \mathrm{h}_p^U(u)\} = \inf\{\mathrm{h}_p^C(h), \mathrm{h}_p^C(f)\} \leq \mathrm{h}_p^C(h)$$

as it follows from the purities of direct summands an from u being divisible. If we allow f to range over F, the equality can be reached when $f = 0$. Thus, $\mathrm{h}_p^C(h) = \sup_{l \in F_t} \mathrm{h}_p^C(x+l)$. We conclude that $\mathrm{h}_p^{C/F_t}(x) = \sup_{l \in F_t} \mathrm{h}_p^C(h+l)$. Note that the proof stays the same if $x \in U$ or even if we replace F_t by $F_t \oplus U$.

Now let $F_t = \bigoplus_{j<i} X_j$, and let $t+1$ be the stage at which one extra finite X_i was introduced. We must show that $F_{t+1} = \bigoplus_{j \leq i} X_j$ and also that F_{t+1} detaches as a direct summand of C. Recall Case 2 of substage II. We had x' such that $px' \in D_t$, $x' \notin D_t$, and

$$m' = \mathrm{h}_p^C(x') = \sup_{d' \in D_s} \mathrm{h}_p^C(x'+d')$$

which is equal, by the previous, to $\mathrm{h}_p^{C/F_s}(x')$. If $px' = u \neq 0$, then

$$m' + 1 \leq \mathrm{h}_p^C(u) = \mathrm{h}_p^C(f),$$

where f is the projection of u onto F_t. If $f = 0$ was the case, we delayed the stage until the required divisible element y of needed order appeared. We know, by the previous lemma, that we did not have to wait forever. If $f \neq 0$, then by the purity of the direct summand F_t in C, and because all elements witnessing finite heights are always put into the V_j at once, we were able to find the needed witness y, but possibly we had to again wait for the divisible part to be sufficiently expanded.

In both cases we obtain a finite cyclic subgroup of C such that $D_{t+1} = X_i \oplus D_t$ is pure in C. Whence, D_{t+1} detaches as a summand of C. $\qquad\square$

Lemma 3.5. *The group $D = \sum_t D_t$ coincides with C.*

Proof. The proof of the Lemma 3.4 together with Lemma 3.3 imply that we are never stuck during the enumeration of D. Suppose there exists an element $g \neq 0$ of C such that $g \notin D = \sum_t D_t$. There exists a number m such that $p^m g \in D$ (e.g, p^m the order of g). Choose m least having this property. Then $g' = p^{m-1}g \notin D$ and $pg' \in D$. We again may assume that the number of g' is the least possible. Then at a stage t of the construction at which pg' was already present in D_t we would have enumerated pg' into the span of D_{t+1}, a contradiction. $\quad\square$

We have constructed, effectively in h_p, a complete decomposition of C into cyclic and quasi-cyclic summands. As we have already mentioned above, it gives us (1) of the theorem.

We prove (2) of the theorem. If A is effectively Δ_2^0-categorical, then we could obtain a Δ_2^0 index for h_p using the isomorphism and the index for h_p in the "regular" copy B of A. Now suppose an index for h_p can be uniformly computed from the index of a computable copy of A. The construction in the proof of (1) is uniform in h_p. We can build an isomorphism to the standard copy of A uniformly in h_p. $\qquad\square$

The next result relates multicyclic groups to equivalence structures. To state the result, we need a definition:

Definition 3.6. For a multicyclic group $A = \bigoplus_{i \in \omega} A_i$, define an equivalence structure E_A by the rule $\#[i] = m$ if $A_i \cong \mathbb{Z}_{p^m}$ and $\#[i] = \infty$ if $A_i \cong \mathbb{Z}_{p^\infty}$.

In the following, we restrict ourselves to non-degenerate multicyclic groups (i.e., having infinitely many quasi-cyclic summands). Notice that A is computably presentable if, and only if, E_A is computably presentable. Recall that the latter is equivalent to the sizes of finite classes being a Σ_2^0 (multi)set[3].

Theorem 3.7. *Let A be computable non-degenerate multicyclic group. Then A is effectively Δ_2^0-categorical if, and only if, E_A is effectively Δ_2^0-categorical.*

Proof. Suppose A is effectively Δ_2^0-categorical. We can pass from a computable copy of E_A to a computable copy of A in which every summand corresponds to an equivalence class. By Theorem 3.1, we can compute an index for h_p, and then use it to compute sizes of classes in the computable copy of E_A. It remains to apply Proposition 2.1.

Now let E_A be an effectively Δ_2^0-categorical equivalence structure. For a set $X \subseteq \omega$, let $X_{-n} = \{m - n : m \in X \text{ and } m \geq n\}$.

Lemma 3.8. *Suppose X is effectively categorical. Then for every n the set $X \cup \bigcup_{i \leq n} X_{-i}$ is effectively categorical. Furthermore, the index for the universal guessing function for $X \cup \bigcup_{i \leq n} X_{-i}$ given by Lemma 2.3 can be obtained uniformly from the index of the universal guessing function for X.*

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[3]If the number of quasi-cyclic summands is finite, then the remark is still valid, but this time both properties are equal to the Σ_2^0 (multi)set being limitwise monotonic – an observation due to Khisamiev [19].

Proof. Given a partial computable φ_e, define for every $0 < k \le n$ a primitive recursive function s by the rule $\varphi_{s(k,e)}(i) = 0$ for $i < k$ and $\varphi_{s(e)}(x+k) = \varphi_e(x)$ if $\varphi_e(x) \downarrow$. Let $r_{e,0} = \#range\,\varphi_e$ and $r_{e,i} = \#range\,\varphi_{s(k,e)}$ for $0 < k \le n$. Let g a be the function for X guessing sizes of ranges of partial recursive functions given by Lemma 2.3.

We define a guessing function d for $X \cup \bigcup_{i \le n} X_{-i}$. Notice that the $r_{e,i}$ are either all finite or infinities all at once. Also, if $r_{e,0} \in X \cup \bigcup_{i \le n} X_{-i}$, then one of the $r_{e,i}$ has to be in X. If $r_{e,i}$ are all undefined, we keep $d(e)$ undefined. If g returns fin for *some* of the $r_{e,i}$, then either $r_{e,i}$ is finite and is outside X or is finite and belongs to X. In both cases we can safely declare $r_{e,0}$ finite. If g tells that some of the $r_{e,i}$ are infinite, and some of them are still undefined, we claim that we can keep $d(e)$ undefined until g either declares one of the $r_{e,i}$ is finite (in which case we set $d(e) = fin$) or tells that all the $r_{e,i}$ are infinite (then we set $d(e) = \infty$). If all of the $r_{e,i}$ are indeed finite but outside X, then we will either keep $d(e)$ undefined, or set $d(e)$ equal to ∞ if all guesses are ∞, or make $d(e) = fin$ if at least one of the guesses is fin. It does not matter since $r_{e,0} \notin X \cup \bigcup_{i \le n} X_{-i}$. If $r_{e,0}$ (and thus, all the $r_{e,i}$) are in fact ∞'s, then g must return ∞ for every $r_{e,i}$. If the classes are finite and $r_{e,0}$ belongs to $X \cup \bigcup_{i \le n} X_{-i}$, then $r_{e,i} \in X$ for some i, and g will eventually converge and tell that $r_{e,i}$ is finite. $\qquad\square$

Let C be a computable copy of A. Define $C(p^n) = \{c \in C : p^n c = 0, p^{n-1} c \ne 0\}$. Notice that $C = \bigcup_n C(p^n)$. We may view cyclic and quasi-cyclic groups as Z_p-modules:

$$\mathbb{Z}_{p^\alpha} \cong \langle a_i : pb_0 = 0; b_{i-1} = pb_i, 0 < i < \alpha \rangle,$$

where $\alpha \in \omega + 1$. We use b with subscripts to denote generators of various elementary summands. Each element in $C(p^n)$ is of the form $\sum_i m_i b_i$, $m_i \in Z_p$, where the elements b_i are generators taken from different components in some fixed complete decomposition of C satisfying $p^n b_i = 0$. We have $p^{n-1} b_i \ne 0$ for at least one i. Then the height of $\sum_i m_i b_i$ is equal to the infimum of heights of the b_i, those including heights of b_j of orders p^m for $m < n$. Let $X = \{m : \mathbb{Z}_{p^m} \text{ is a summand of A}\}$. Clearly $\{h_p^C(b_i) : p^n b_i = 0, p^{n-1} b_i \ne 0\} \cap \omega = X_{-n}$, where b_i are the generators of the elementary components from the fixed decomposition of C. Since the height of $\sum_i m_i b_i$ is equal to the infimum of heights of the b_i, we have

$$\{h_p^C(c) : p^n c = 0, p^{n-1} c \ne 0\} \cap \omega \subseteq X \cup \bigcup_{m \le n} X_{-m}.$$

We show that these sets are equal. Since quisi-cyclic summands are present in C, we may take any b_j of order $p^m < p^n$ and consider $b_j + b_i$, where b_i is of order p^n and has infinite height. The height of $b_j + b_i$ is equal to the height of b_j. Therefore,

$$\{h_p^C(c) : p^n c = 0, p^{n-1} c \ne 0\} \cap \omega = X \cup \bigcup_{i \le n} X_{-n}.$$

For every n, we construct an equivalence structure as follows. For every non-zero $c \in C(p^n)$, construct a class of size equal to $h_p^C(c)$. Denote the resulting equivalence structure by E_n. We have $E_n \cong E(X \cup \bigcup_{m \le n} X_{-m})$, where $X = \{m : \mathbb{Z}_{p^m} \text{ is a summand of A}\}$. By the choice of X and Lemma 3.8, there is a $0'$-effective procedure uniform in n which guesses sizes of equivalence classes in E_n correctly. Using this procedure, we can compute $h_p^C(c)$. It remains to apply Theorem 3.1. $\qquad\square$

Our results have several interesting corollaries. The first corollary is a consequence of Theorems 3.7 and 2.5:

Corollary 3.9. *Effective Δ_2^0-categoricity of a non-degenerate multicyclic group A is completely regulated by the set $X = \{m : \mathbb{Z}_{p^m} \text{ is a summand of } A\} \subseteq \omega$.*

In other words, repetitions of finite summands do not matter. Notice that proving this fact directly, without any reference to equivalence structures, would require a lot more effort. The second corollary answers a question left open in [5]:

Corollary 3.10. *There exists a computable multicyclic group A in which orders of finite summands are unbounded and which is Δ_2^0-categorical (and is in fact effectively Δ_2^0-categorical).*

Proof. Follows from Theorems 1.4, 2.5, and 3.7. □

Notice also that the combination of Theorems 1.4, 2.5, and 3.7 gives a lot more information about (effectively) Δ_2^0-categorical multicyclic groups.

4. Bounding effective categoricity is the same as completeness

In this section we prove the second main result of the paper. The proofs contained in this section heavily rely on Lemma 2.3 and Remark 2.4.

Definition 4.1. We say that a function f is *strongly dominant* if there is $h \leq_T \emptyset'$ such that for each total φ_e, we have $f(x) > \varphi_e(x)$ for every $x > h(e)$. Note: if we replace "$h \leq_T \emptyset'$" by "$h \leq_T f$" then this is equivalent to $f \geq_T \emptyset'$. A degree is strongly dominant if it computes a strongly dominant function.

Theorem 4.2. *For a c.e. degree $\boldsymbol{a}$, the following are equivalent:*

  (1) $\boldsymbol{a} = \boldsymbol{0}'$.
  (2) $\boldsymbol{a}$ is strongly dominant.
  (3) There exists some infinite set $X \leq_T \boldsymbol{a}$ such that X is effectively categorical.

Proof. (1) $\Rightarrow$ (3): Each strongly dominant degree computes an infinite Δ_2^0 categorical set. Hence a strongly dominant degree does not form a minimal pair with $\emptyset'$. Is this interesting?

(2) $\Rightarrow$ (1): This is Theorem 4.8.

(3) $\Rightarrow$ (2): Fix a c.e. set A and a Turing functional Φ such that $X = \Phi^A$ is infinite and effectively categorical via the function $g \leq_T \emptyset'$. That is, $g : \omega \mapsto \{f, \infty\}$ satisfies Lemma 2.3(ii) via Remark 2.4. The enumeration of A and the functional Φ induces, in the natural way, an approximation $\{X_s\}_{s \in \omega}$ of the set X. Define the sequence $\{z_i\}_{i \in \omega}$ by the following. Let z_{i+1} be the largest number larger than z_i such that for some s, we have $X_s(j) = 0$ for every $z_i < j < z_{i+1}$ and $X_s(z_{i+1}) = 1$. This sequence exists because X is infinite, and can be computed using A (and the enumeration of A). Now define $f(i)$ to be the least stage s such that $X \restriction z_i + 1[s] = \Phi^A \restriction z_i + 1[s]$ converges correctly on all inputs up to and including z_i. Clearly $f \leq_T A$.

Define the computable function p such that for each e, $\varphi_{p(e)}$ is the following function. Define s_x to be the first stage larger than s_{x-1} such that $\varphi_e(x)[s_x] \downarrow$. At each stage s_x check if currently we have $g(p(e)) = \infty$ and $\varphi_{p(e)}(y-1) \in X$, where y is the least input for which $\varphi_{p(e)}$ is not yet defined. If so do nothing at this stage, otherwise define $\varphi_{p(e)}(y) \downarrow = y'$ where and y' is the least number larger than $\varphi_{p(e)}(y-1)$ which is currently in X_{s_x}. Note that the Recursion Theorem is used here. Now let $h(e) = x + 1$ where s_x is the largest stage of this form which is less than or equal to the stage t after which $g_t(p(e))$ is stable. Clearly $h \leq_T \emptyset'$.

Claim 4.3. *Fix i, y, s, and let $y' > y$ be the least number in A_s larger than y. If $z_i \geq y$ then we also have $z_{i+1} \geq y'$.*

Proof of claim. Since $z_{i+1} > z_i$ we may assume that $y \leq z_i < y'$. Now apply the definition of z_{i+1}. □

Now fix e such that φ_e is total. We show that $f(x) > \varphi_e(x)$ for every $x > h(e)$. Observe that $g(p(e)) = \infty$, since s_x exists for every x. It follows easily that we must have that $\varphi_{p(e)}$ is total.

Claim 4.4. *For every x, we have that at (the end of) stage s_x, $\varphi_{p(e)}(y) \leq z_x$, where y is the maximum element in $dom\varphi_{p(e)}$.*

Proof of claim. We proceed by induction on x. At stage s_0 the construction will define $\varphi_{p(e)}(0)$ to be the first element of X_{s_0}. Clearly $z_0 \geq \varphi_{p(e)}(0)$, by the definition of z_0. Apply Claim 4.3 for the inductive step. $\square$

Now we assume that there is a least number $x > h(e)$ so that $f(x) \leq \varphi_e(x)$. This means that A is stable on the use of $\Phi^A \upharpoonright z_x + 1$ after stage s_x. By Claim 4.4 we have that $X \upharpoonright \max dom\varphi_{p(e)}$ is stable after stage s_x. By the definition of $h(e)$ we must have that $g_s(p(e)) = \infty$ for every $s \geq s_x$. Hence it follows that $dom\varphi_{p(e)}$ never grows after s_x, a contradiction. $\square$

An infinite Σ_2^0 set is (effectively) categorical if the corresponding non-degenerate equivalence structure $E(X)$ is (effectively) Δ_2^0-categorical.

Corollary 4.5. *Below each high incomplete c.e. degree there exists an infinite set X which is categorical but not effectively categorical.*

Proof. Follows from the theorem above, and from the second main result of the satellite paper [7]. $\square$

To prove Theorem 4.8, we accumulate more knowledge about effectively categorical sets. The lemma below may also be of some independent interest to the reader.

Lemma 4.6. *If a degree $\boldsymbol{a} \leq \boldsymbol{0}'$ is strongly dominant then there is some partial $\boldsymbol{a}$-computable function F and some total function $G \leq_T \emptyset'$ such that $J^{\emptyset'} = F \circ G$. (Here $J^{\emptyset'}(n) = \Phi_n^{\emptyset'}(n)$.)*

Proof. Suppose $f \leq_T \boldsymbol{a}$ is strongly dominant via the function $g \leq_T \emptyset'$. Fix a computable approximation $\{f_s\}$ of f, i.e. $\lim_s f_s(x) = f(x)$ for every x. Define a stage s to be *bad for n* if $\emptyset'$ changes at stage s below the use of an existing computation $J^{\emptyset'}(n)[s-1]$.

Let p be a computable function (given by the Recursion Theorem) such that for every n and x, we search for the first stage $t > x$ such that $\varphi_{p(n)}(z)[t] \downarrow$ for every $z < x$ and one of the following holds:

    (i) $g(p(n))[t] \geq x$,
    (ii) t is a bad stage for n, or
    (iii) there is some z and b with $g(p(n))[t] < z \leq b \leq x$ such that b is a bad stage for n and $f_t(z) \leq \varphi_{p(n)}(z)$.

If t is found we set $\varphi_{p(n)}(x) = \max\{f_u(x) \mid u \leq t\}$.

Claim 4.7. *Suppose that $b > g(p(n))$ is a bad stage for n. Then for every $g(p(n)) < z \leq b$ we have $\varphi_{p(n)}(z) \downarrow < f(z)$. Furthermore $\varphi_{p(n)}$ is total iff there are infinitely many bad stages for n.*

Proof of claim. The fact that $\varphi_{p(n)}(z) \downarrow$ follows from (ii) of the construction. Now suppose that $f(z) \leq \varphi_{p(n)}(z)$. If $f_t(z)$ falls below $\varphi_{p(n)}(z)$ then (iii) in the construction would ensure that $\varphi_{p(n)}$ gets defined on more and more inputs. If $f_t(z)$ remains permanently smaller than $\varphi_{p(n)}(z)$ then $\varphi_{p(n)}$ would end up being a total function and we get a contradiction since $z > g(p(n))$.

Now suppose there are only finitely many bad stages. Let b be the largest bad stage. Since $\varphi_{p(n)}(z) < f(z)$ for every $g(p(n)) < z \leq b$ we will eventually stop issuing definitions for $\varphi_{p(n)}$ once f_t is stable below $b + 1$. Hence $\varphi_{p(n)}$ is total iff there are infinitely many bad stages. $\square$

Now define $F(\langle n, x \rangle)$ by the following. Find the first pair $z < y$ larger than x such that $J^{\emptyset'}(n)[z] \downarrow$, there is no bad stage $z < b \leq y$ and where $\varphi_{p(n)}(y)$ has not yet received a definition at stage t where t is the first stage such that $f_t(y) = f(y)$. If such a pair is found we set $F(\langle n, x \rangle) = J^{\emptyset'}(n)[z]$. Clearly F is $\boldsymbol{a}$-partial computable. Now define $G(n) = \langle n, g(p(n)) \rangle$.

Fix n. Suppose that $F(G(n)) \downarrow$. Let $g(p(n)) < z < y$ be the corresponding pair. There cannot be a bad stage $b > y$ because otherwise we must have $\varphi_{p(n)}(y) \downarrow$. Since this definition must be issued by the construction after stage t, we must have $\varphi_{p(n)}(y) > f_t(y) = f(y)$. This contradicts Claim 4.7. Hence there cannot be a bad stage $b > y$. Hence $J^{\emptyset'}(n)[z] = J^{\emptyset'}(n) \downarrow$. This means that if $J^{\emptyset'}(n) \uparrow$ then $F(G(n)) \uparrow$.

Now suppose that $J^{\emptyset'}(n) \downarrow$. Fix z_0 to be any number larger than $g(p(n))$ where the computation $J^{\emptyset'}(n)[z_0]$ is stable. There must be some number $y_0 > z_0$ such that $\varphi_{p(n)}(y) \uparrow$ (since the domain of $\varphi_{p(n)}$ is closed downwards and $\varphi_{p(n)}$ is not total). The pair $z_0 < y_0$ means that $F(G(n))$ must be defined, say with a corresponding pair $z < y$. By the same argument in the preceding paragraph, we have $F(G(n)) = J^{\emptyset'}(n)[z] = J^{\emptyset'}(n)$. $\square$

Theorem 4.8. *Let $\boldsymbol{a} \leq \boldsymbol{0}'$. Then $\boldsymbol{a}$ is strongly dominant iff $\boldsymbol{a} = \boldsymbol{0}'$.*

Proof. The right to left direction is trivial, so we assume that the set A has strongly dominant degree. By Lemma 4.6 we fix a Turing functional Ψ and a computable function $g(x, s)$ such that $J^{\emptyset'} = \Psi^A \circ G$, where $G(x) = \lim_s g(x, s)$. Since $A \leq_T \emptyset'$ we let $\Psi^A(x)[s]$ be a computable approximation to $\Psi^A(x)$ at stage s; this can of course be divergent. We now define a computable sequence $\{p_m(x, s)\}$ of computable functions such that $P_m(x) = \lim_s p_m(x, s)$ exists for every m and x. By the Recursion Theorem there is a computable function $q(m)$ such that $P_m(q(m)) = J^{\emptyset'}(q(m))$ for every m.

Defining the sequence $\{p_m(x, s)\}$. We set $p_m(x, 0) = 0$ for every m, x. At stage $s + 1$ we set $p_m(x, s + 1) = s + 1$ if $m \in \emptyset'_{s+1} - \emptyset'_s$ or if $g(q(m + 1), s + 1) \neq g(q(m + 1), s)$. Otherwise set $p_m(x, s + 1) = p_m(x, s)$. It is clear that $P_m(x) = \lim_s p_m(x, s)$ exists for every m and x. We define the sequence $\{s_m\}$ by the following. Let s_{-1} be the first stage such that $g(q(0), -)$ is stable. Inductively define s_m to be the least stage larger than s_{m-1} such that $\Psi^A(g(q(m), s_m))[s_m] \downarrow$ where the use of the computation $\Psi^A(g(q(m), s_m))[s_m]$ is correct. (Note that A is used to find the stage s_m for $m \geq 0$, and each s_m exists because $P_m(q(m))$ is convergent).

Claim 4.9.
  (i) $g(q(m), t) = g(q(m), s_m)$ *for every $t \geq s_m$.*
  (ii) $p_m(q(m), -)$ *is never increased after stage s_m.*

Proof of claim. For $m = 0$ we automatically have (i) since $s_0 > s_{-1}$. (ii) also follows because

$$(1) \qquad\qquad s_m > \Psi^A(g(q(m), s_m))[s_m] = \Psi^A(G(q(m)) = J^{\emptyset'}(q(m)).$$

The first inequality above holds because we adopt the usual convention: any computation which converges in t steps will have output less than t. Now if (i) and (ii) holds for $m - 1$ then we easily see that (i) holds for m; otherwise if (i) fails then the construction will increase $p_{m-1}(q(m - 1), -)$ after stage $s_m > s_{m-1}$. To see that (ii) holds for m note that Equation 1 holds for m as well. $\square$

Finally to see that $\emptyset' \leq_T A$ we claim that $m \in \emptyset'$ if and only if $m \in \emptyset'[s_m]$. This follows from Claim 4.9(ii). $\square$

We now show a basis theorem for perfect Π^0_1 classes. A consequence of this is that there exists a strongly dominant degree which is generalized low. Hence outside of the Δ^0_2 degrees strong domination and computing $\emptyset'$ are not the same.

Theorem 4.10. *Each perfect Π_1^0 class P contains a path A which is of strongly dominant degree.*

Proof. Fix a perfect Π_1^0 class P. It is easy to see that P contains a member of high degree: Let $T : 2^{<\omega} \mapsto 2^{<\omega}$ be a function tree such that $[T] = P$. Recall that a node σ is extendible in P if $[\sigma] \cap P \neq \emptyset$. Since P is perfect, to define T we only need to remove all non-extendible nodes of P. Hence $T \leq_T \emptyset'$. Now observe that if $C \geq_T \emptyset''$ then $T(C)$ is high, because $C \leq_T T(C) \oplus T \leq_T T(C)'$.

We now argue that $A = T(C)$ has strongly dominant degree, where we take $C = \{e \mid \varphi_e$ is total$\}$. Since the set of non-extendible nodes of P is c.e., we fix a computable approximation $\{T_s\}$ of T where for each s, T_s is a computable function tree which contains all the currently extendible nodes of P (at stage s).

Now define $f(n)$ by the following. For each $i \leq n$ we let σ_i be a string of length i such that $A \supset T_n(\sigma_i)$. If σ_i exists and $\sigma_i(|\sigma_i|) = 1$ we wait for either T to change below level $i + 1$ or for $\varphi_i(n) \downarrow$. Define $f(n)$ to be the largest number seen.

It is easy to check that f is total: If we get stuck at some i while waiting to define $f(n)$ then we must have $i \in C$ and so φ_i is total. Clearly we also have $f \leq_T A$. Now define $h(n)$ to be the least stage s after which T_s is stable on the first $n + 1$ levels. We have $h \leq_T \emptyset'$.

Now take i such that φ_i is total. Let $n > h(i)$. By the definition of f we must have $f(n) > \varphi_i(n)$. Hence f is strongly dominant via h. $\square$

Corollary 4.11. *There is a strongly dominant degree which is GL_1. Hence not every strongly dominant degree is Turing complete.*

Proof. There is a special Π_1^0 class where every path is GL_1. $\square$

Thus, outside of the Δ_2^0 degrees, strong dominance does not guarantee completeness according to Corollary 4.11.

5. Further topics

It would be nice to have an answer to:

Question 5.1. *Does Theorem 3.7 hold for plain Δ_2^0-categoricty?*

We conjecture that there exists a non-degenerate multicyclic group which is Δ_2^0-categorical but not effectively so. We also expect that some of the results in the paper can be lifted to higher Ulm types.

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